Bulking I: an Abstract Theory of Bulking

Marianne Delorme, Jacques Mazoyer, Nicolas Ollinger, Guillaume Theyssier

Bulking I: an Abstract Theory of Bulking ⋆

M. Delorme [a], J. Mazoyer [a], N. Ollinger [b,1], G. Theyssier [c]

[a] *LIP, ENS Lyon, CNRS, 46 allée d'Italie, 69 007 Lyon, France*

[b] *LIF, Aix-Marseille Université, CNRS, 39 rue Joliot-Curie, 13 013 Marseille, France*

[c] *LAMA, Université de Savoie, CNRS, 73 376 Le Bourget-du-Lac Cedex, France*

Abstract

This paper is the first part of a series of two papers dealing with bulking: a quasi-order on cellular automata comparing space-time diagrams up to some rescaling. Bulking is a generalization of grouping taking into account universality phenomena, giving rise to a maximal equivalence class. In the present paper, we discuss the proper components of grouping and study the most general extensions. We identify the most general space-time transforms and give an axiomatization of bulking quasi-order. Finally, we study some properties of intrinsically universal cellular automata obtained by comparing grouping to bulking.

Key words: cellular automata, bulking, grouping, classification

Bulking is introduced as a tool to structure cellular automata, considered as the sets of their orbits. To achieve this goal, sets of orbits are considered up to spatio-temporal transforms. Such quotients are then compared according to algebraic relations to obtain quasi-orders on the set of cellular automata, in a way similar to reductions in the case of recursive functions. It turns out that the obtained equivalence classes tend to capture relevant properties: in particular, the greatest element, when it exists, corresponds to a notion of intrinsic universality. The first and present paper is concerned with the choice of the main ingredients to define an interesting bulking. The second paper, *Bulking II: Classifications of Cellular Automata* [4], studies the structure of the main three varieties of bulking.

A cellular automaton is a discrete dynamical system consisting of a network of cells fulfilling the following properties: each cell acts as a finite state machine;

⋆ The results presented here first appeared to a great extent in French in the PhD theses of Ollinger [13] and Theyssier [14]

1 Corresponding author (`Nicolas.Ollinger@lif.univ-mrs.fr`)

the network is regular; interactions are local, uniform and synchronous. A space-time diagram is the geometrical representation of an orbit obtained by pilling up the successive configurations. The present paper aims at structuring the sets of space-time diagrams generated by cellular automata.

To deal with the richness of these objects, some families of space-time diagrams are identified through spatio-temporal transforms preserving the notion of cellular automata. In particular, one type of transform, commonly used in algorithmic constructions in the cellular automata literature, is to be taken into account: cells grouping.

A very common grouping transform appears early in algorithmic constructions on cellular automata as a tool to simplify the description of the algorithm. A typical use of this tool appear in the work of Fischer [6]. To recognize prime numbers in real-time, a first construction is given to recognize the primality of n at time $3n$, the cells are then packed in 3×3 blocks defining a new cellular automaton achieving real-time recognition, as depicted on Figure 1.

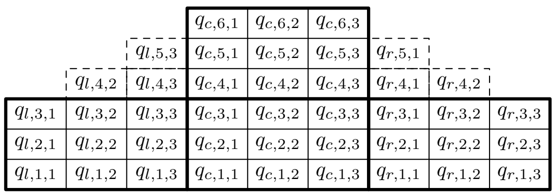


Fig. 1. Grouping cells *à la* Fischer

Rescaling also appears when comparing neighborhoods, in particular the relation between first neighbors and the one-way neighborhood. To simulate, up to a translation, a first neighbors cellular automaton by a one-way cellular automaton, Choffrut and Čulik II [3] proposes to add to the set of states every pair of original states and compute a transition in two time steps, as depicted on Figure 2. Ibarra *et al.* [8] proposes to achieve the simulation in real-time by grouping pairs of consecutive cells in a single cell, as depicted on Figure 3.

Fig. 2. Grouping cells *à la* Choffrut et Čulik II

Fig. 3. Grouping cells *à la* Ibarra *et al.*

Rescaling can be used as a tool to describe a dynamical property like nilpotency, where every configuration evolves in uniform time T into a single fixpoint configuration. Every nilpotent cellular automaton with n states and a transition time T is simulated by a cellular automaton with 2 states and transition time 2 by grouping every T transitions in the first automaton and grouping cells by at least $\lceil \log_2 n \rceil$ in the second automaton, as depicted on Figure 4.

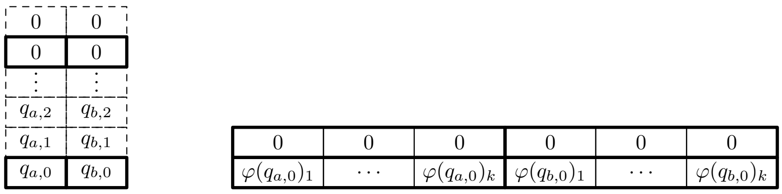


Fig. 4. Grouping cells for nilpotency

A last example of the classical use of rescaling and simulation is the notion of intrinsic universality. A cellular automaton is intrinsically universal if it can simulate, up to rescaling, every cellular automaton. The existence of such cellular automata appears in the work of Banks [2] and more and more precise definitions were successively proposed by Albert and Čulik [1], Martin [10], Durand and Róka [5]. The notion of simulation used in these articles is the following. To simulate a given cellular automaton, each cell of the initial configuration is encoded as a segment of cells of the universal automaton and each transition is simulated by a fixed number of transitions (depending on the chosen encoding and the simulated cellular automaton) as depicted on Figure 5.

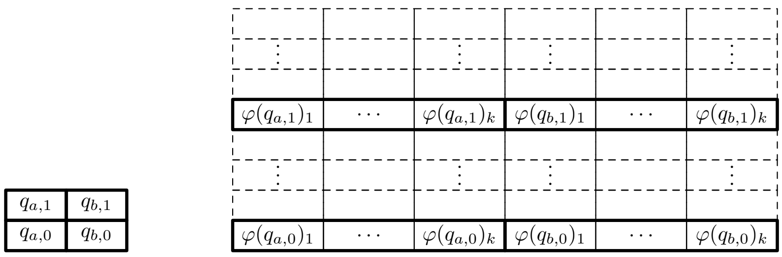


Fig. 5. Grouping cells for intrinsic universality

Bulking generalizes the use of rescaling as a tool to compare cellular automata. Figures 6, 7 and 8 show examples of simulations of some cellular automata by others. For each of them, both geometric transformation of space-time and local transformation on cellular automata rules are shown (time goes from bottom to top in each figure). These examples are intentionally simple, but the simulation relations are studied more in depth in the second paper. All cellular automata chosen in these examples have the same neighborhood (cell itself, left and right immediate neighbors). Their local rules are the following.

Just gliders. Two states interpreted as particules moving left (◁) and right (▷) evolve in a quiescent background state (□). When two opposite particles meet they annihilate, leaving a background state (□).

ECA 184. The line of cells is interpreted as a highway where □ states represent cars and ■ represent free portions of highways. Cars move to the right

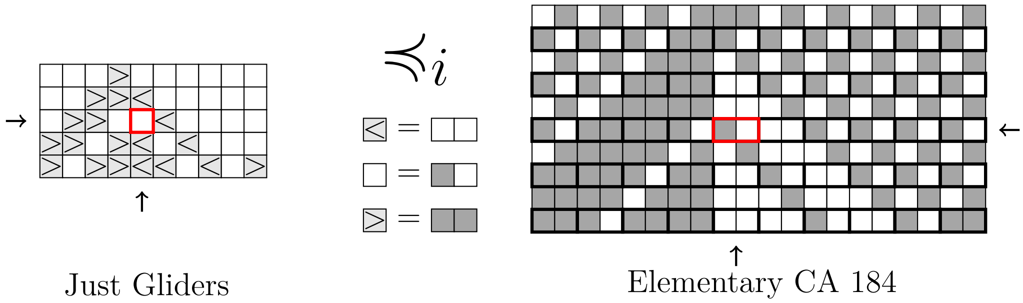


Fig. 6. Injective simulation of ’Just gliders’ by ECA 184.

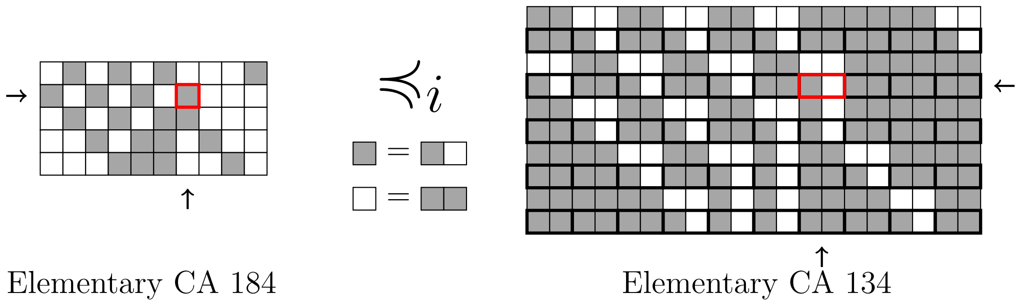


Fig. 7. Injective simulation of ECA 184 by ECA 134.

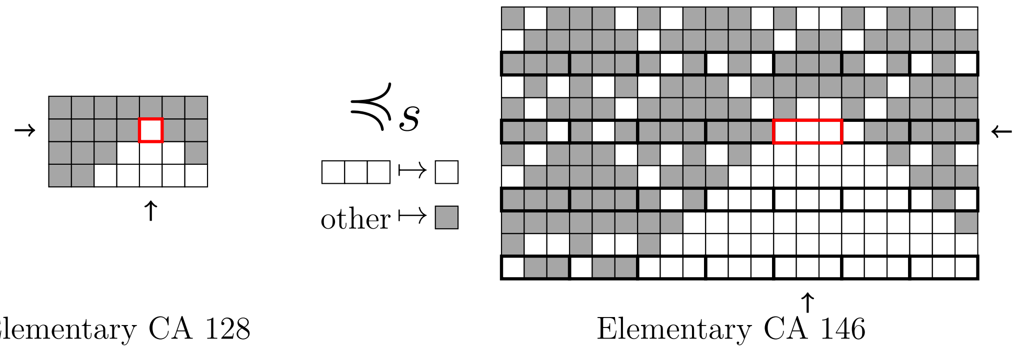


Fig. 8. Surjective simulation of ECA 128 by ECA 146.

by one cell if they can (no car present on the next cell), otherwise they don’t move.

ECA 134. If state □ is interpreted as 1 and ■ as 0 then the rule is simply the addition modulo 2 of neighbouring states, except when the left cell is 1 and others 0.

ECA 128. The state ■ is spreading over the quiescent state □.

ECA 146. If state □ is interpreted as 1 and ■ as 0 then the rule is simply the addition modulo 2 of neighbouring states, except when the central cell is 1 and others 0.

This first paper, discussing the proper components of bulking is organized as follows. In section 1, definitions are given with a geometrical point of view. In section 2, grouping is presented. In section 3, possible extensions are investigated, searching for good candidates of geometrical transforms and elementary simulation relation. In section 4, bulking is defined as a formal family of simulation quasi-orders and an extension of grouping is chosen. In section 5, this extension is compared to grouping and a first result concerning intrinsically universal cellular automata is obtained.

1 Patterns, Colorings and Cellular Automata

1.1 Patterns

To simplify writings and manipulation of parts of space-time diagrams, several notations are introduced in this section and depicted on Figure 9.

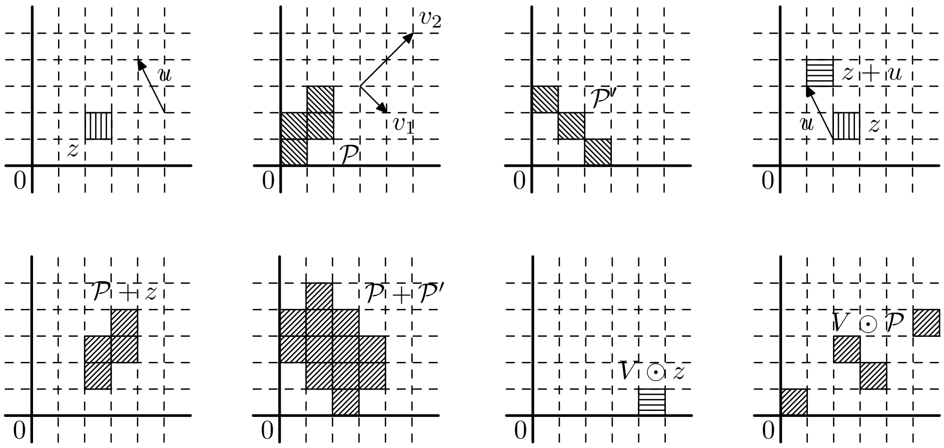


Fig. 9. Geometrical operators

A *pattern* $\mathcal{P}$ is a subset of $\mathbb{Z}^d$. The *m-rectangular pattern* $\boxplus_m$ is the pattern $\{0,\ldots,m_1-1\}\times\cdots\times\{0,\ldots,m_d-1\}$. Using the natural extension of $+$ on sets, the *translation* $\mathcal{P}+u$ of $\mathcal{P}$ by a vector $u\in\mathbb{Z}^d$ is the pattern $\{z+u|z\in\mathcal{P}\}$ and the *sum* of two patterns $\mathcal{P}$ and $\mathcal{P}'$ is the pattern $\{z+z'|z\in\mathcal{P},z'\in\mathcal{P}'\}$. An *elementary translation* is a translation by a vector $\varsigma_k\in\mathbb{Z}^d$ with all coordinates equal to 0 but the kth which is equal to 1 or -1. Every translation is obtained by composition of elementary translations.

A *basis* $V\in\left(\mathbb{Z}^d\right)^d$ is a tuple of d non-zero linearly independent vectors $(v_1,\ldots,v_d)$. The *m-rectangular basis* $\square_m$ is the basis $(m_1\delta_1,\ldots,m_d\delta_d)$ where δ_k has all coordinates equal to 0 but the kth which is equal to 1. The *image* $V\odot z$ by V of a point $z\in\mathbb{Z}^d$ is the point $\sum_{i=1}^d z_iv_i$. The *image* $V\odot\mathcal{P}$ by V of a pattern $\mathcal{P}$ is the pattern $\{V\odot z|z\in\mathcal{P}\}$. The *image* by a basis V' of a basis V is the basis $V'\odot V=(V'\odot v_1,\ldots,V'\odot v_d)$. Notice that $\square_m\odot\square_{m'}=\square_{mm'}$ where for all k, $(mm')_k=m_km'_k$.

A *tiling of space* is a pair $(\mathcal{P},V)$ where $\mathcal{P}$ is a pattern that tiles the plane with the basis V, *i.e.* such that $\left\{\mathcal{P} + V \odot z \middle| z \in \mathbb{Z}^d\right\}$ is a partition of $\mathbb{Z}^d$. Notice that the size of $\mathcal{P}$ has to be $|\det V|$, as depicted on Figure 10. Given a basis V, the equivalence relation $\equiv_V$ on $\mathbb{Z}^d$ is defined by $z \equiv_V z'$ if $z' - z \in V \odot \mathbb{Z}^d$. It defines precisely $|\det V|$ equivalence classes. Valid patterns are precisely patterns consisting of one point in each equivalence class of $\equiv_V$. The *m-rectangular tiling* is the tiling $(\boxplus_m, \square_m)$. The *composition* $(\mathcal{P}', V') \circ (\mathcal{P}, V)$ of two tilings of space $(\mathcal{P}', V')$ and $(\mathcal{P}, V)$ is the tiling of space $(\mathcal{P} + V \odot \mathcal{P}', V' \odot V)$, as depicted on Figure 11. Notice that $(\boxplus_m, \square_m) \circ (\boxplus_{m'}, \square_{m'}) = (\boxplus_{mm'}, \square_{mm'})$.

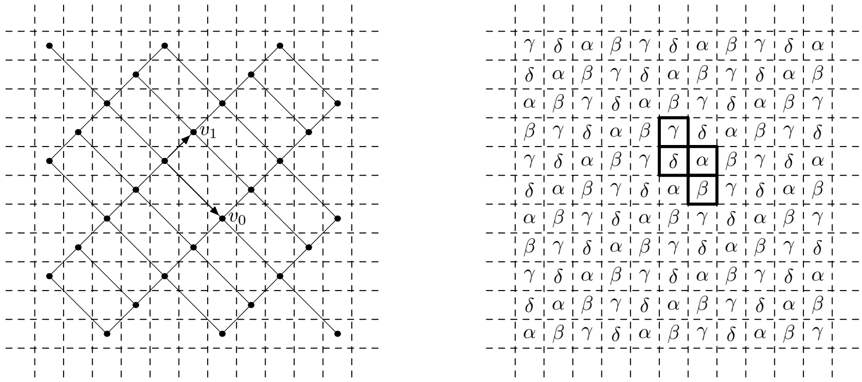


Fig. 10. Network and equivalence classes associated to a family of vectors of $\mathbb{Z}^2$

1.2 Colorings

A *coloring* $\mathcal{C} \in \Sigma^{\mathcal{P}}$ by letters of a finite alphabet Σ is a covering of its support $\mathcal{P} \subseteq \mathbb{Z}^d$, denoted as $\mathrm{Sup}(\mathcal{C})$. A *singleton coloring* is a coloring with a singleton support. A *finite coloring* is a coloring with finite support. A *full coloring* is a coloring with support $\mathbb{Z}^d$. A coloring $\mathcal{C}'$ is a *subcoloring* of a coloring $\mathcal{C}$, denoted as $\mathcal{C}' \ll \mathcal{C}$, if $\mathcal{C}'$ is a restriction of $\mathcal{C}$, *i.e.* $\mathcal{C}' = \mathcal{C}_{|\mathrm{Sup}(\mathcal{C}')}$. The *translation* $u \cdot \mathcal{C}$ of $\mathcal{C}$ by a vector $u \in \mathbb{Z}^d$ is the coloring with support $\mathrm{Sup}(\mathcal{C}) + u$ satisfying, for all $z \in \mathrm{Sup}(\mathcal{C})$, $u \cdot \mathcal{C}(z + u) = \mathcal{C}(z)$. The *u-shift* is the translation map over full colorings $\sigma_u : \Sigma^{\mathbb{Z}^d} \to \Sigma^{\mathbb{Z}^d}$ defined for all coloring $\mathcal{C}$ by $\sigma_u(\mathcal{C}) = u \cdot \mathcal{C}$. An *elementary shift* is a shift by an elementary translation. A coloring $\mathcal{C}$ occurs in a coloring $\mathcal{C}'$, denoted as $\mathcal{C} \Subset \mathcal{C}'$ if some translation of $\mathcal{C}$ is a subcoloring of $\mathcal{C}'$. A coloring $\mathcal{C}$ is *periodic*, with periodicity vector $u \in \mathbb{Z}^d$, if for all $z \in \mathrm{Sup}(\mathcal{C}) \cap (\mathrm{Sup}(\mathcal{C}) - u)$, $\mathcal{C}(z) = \mathcal{C}(z + u)$. Given a color $s \in \Sigma$, a coloring $\mathcal{C}$ is *s-finite* if $\mathcal{C}$ is equal to s everywhere but on a finite support.

The *cylinder* generated by a coloring $\mathcal{C}$ over an alphabet Σ is the set of full colorings $[\mathcal{C}] = \left\{\mathcal{C}' \in \Sigma^{\mathbb{Z}^d} \middle| \mathcal{C} \ll \mathcal{C}'\right\}$. The *Cantor topology* on $\Sigma^{\mathbb{Z}^d}$ is the product topology of the discrete topology on Σ. Its open sets are generated by the cylinders of finite colorings. This topology is compact, metric and perfect [7].

A packing map transforms a coloring into another coloring by packing together cells according to a given tiling. Formally, the *packing map* with tiling $(\mathcal{P}, V)$

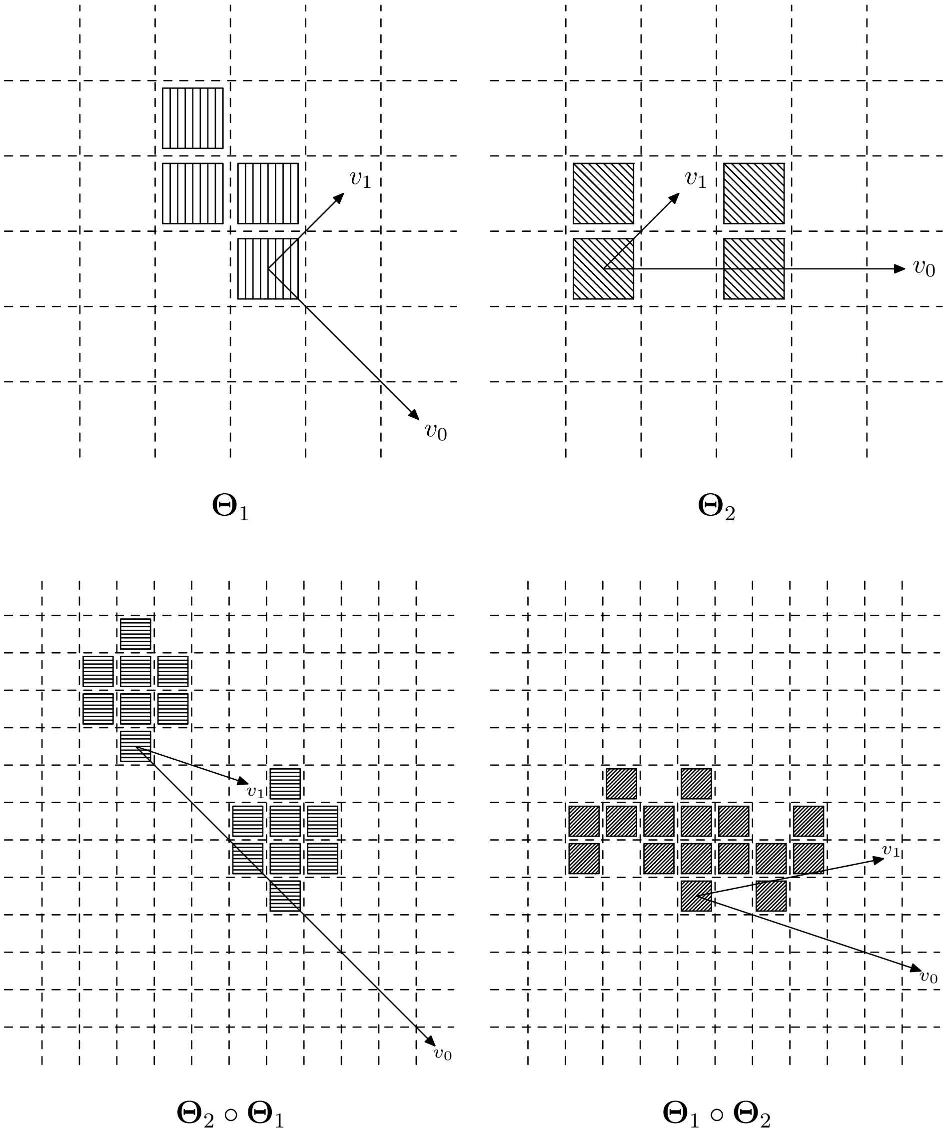


Fig. 11. Composition of two tilings of the plane

over the alphabet Σ is the map $\langle \mathcal{P}, V \rangle : \Sigma^{\mathbb{Z}^d} \to (\Sigma^{\mathcal{P}})^{\mathbb{Z}^d}$ defined for all full colorings $\mathcal{C} \in \Sigma^{\mathbb{Z}^d}$ and all points $z \in \mathbb{Z}^d$ by $\langle \mathcal{P}, V \rangle (\mathcal{C})(z) = ((-V \odot z) \cdot \mathcal{C})_{|\mathcal{P}}$. The *rectangular packing map* $\square^m$ is the packing map $\langle \boxplus_m, \square_m \rangle$, its inverse is denoted as $\square^{-m}$.

1.3 Cellular Automata

A *d-dimensional cellular automaton* (d-CA) $\mathcal{A}$ is a triple (S, N, f) where S is a finite set of states, N is the neighborhood, a finite pattern of $\mathbb{Z}^d$ and $f : S^N \to S$ is the local rule of $\mathcal{A}$. A *configuration* of $\mathcal{A}$ is a mapping $c \in S^{\mathbb{Z}^d}$. The *global transition function* $G : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ of $\mathcal{A}$ maps every configuration $c \in S^{\mathbb{Z}^d}$ to the configuration $G(c) \in S^{\mathbb{Z}^d}$ such that, for all $z \in \mathbb{Z}^d$, $G(c)(z) = f((-z \cdot c)_{|N})$. The *space-time diagram* of $\mathcal{A}$ starting from a configuration c_0 is a mapping $\Delta \in S^{\mathbb{N}\times\mathbb{Z}^d}$ encoding an infinite sequence of successive orbits of the dynamical system $\left(S^{\mathbb{Z}^d}, G\right)$ by $\Delta(0) = c_0$ and, for all $t \in \mathbb{Z}^+$, $\Delta(t) = G(\Delta(t-1))$. The set of space-time diagrams of a CA $\mathcal{A}$ is denoted as $\operatorname{Diag}\mathcal{A}$. We will call *autarkic* any d-CA with neighborhood $\{0\}$. Every d-CA with a singleton neighborhood is the composition of a shift by an autarkic CA.

A d-CA $\mathcal{A}$ is a *subautomaton* of a d-CA $\mathcal{B}$, with respect to the injective map $\varphi : S_{\mathcal{A}} \to S_{\mathcal{B}}$, denoted as $\mathcal{A} \sqsubseteq_\varphi \mathcal{B}$, if $G_{\mathcal{B}} \circ \overline{\varphi} = \overline{\varphi} \circ G_{\mathcal{A}}$ where $\overline{\varphi} : S_{\mathcal{A}}^{\mathbb{Z}^d} \to S_{\mathcal{B}}^{\mathbb{Z}^d}$ is the canonical extension of φ defined for all $c \in S^{\mathbb{Z}^d}$ by $\overline{\varphi}(c) = \varphi \circ c$. Equivalently stated, a d-CA $\mathcal{A}$ is a subautomaton of a d-CA $\mathcal{B}$ with respect to φ if and only if $\overline{\varphi}(\operatorname{Diag}\mathcal{A}) \subseteq \operatorname{Diag}\mathcal{B}$. A d-CA $\mathcal{A}$ is *isomorphic* to a d-CA $\mathcal{B}$, denoted as $\mathcal{A} \equiv \mathcal{B}$, if both $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{B} \sqsubseteq \mathcal{A}$.

In this paper, we focus on CA seen as discrete dynamical systems, that is the pair $(S^{\mathbb{Z}^d}, G)$ up to isomorphism and more precisely the orbits of such systems (represented by space-time diagrams). Thanks to the following theorem, we can freely manipulate global rules of CA to generate new CA.

Theorem 1 (Hedlund [7]) *A map $G : \Sigma^{\mathbb{Z}^d} \to \Sigma^{\mathbb{Z}^d}$ is the global transition function of a cellular automaton if and only if G is continuous and commutes with elementary translations.*

A CA is injective (*resp.* surjective, bijective) if its global rule is injective (*resp.* surjective, bijective). By previous theorem, the composition of two CAs, the cartesian product of two CAs or the inverse of a bijective CA is a CA. The *cartesian product* of two d-CA $\mathcal{A}$ and $\mathcal{B}$ is the d-CA $\mathcal{A} \times \mathcal{B}$ whose global transition function verifies for all $(c, c') \in S_{\mathcal{A}}^{\mathbb{Z}^d} \times S_{\mathcal{B}}^{\mathbb{Z}^d}$, $G_{\mathcal{A}\times\mathcal{B}}((c, c')) = (G_{\mathcal{A}}(c), G_{\mathcal{B}}(c'))$. A *reversible cellular automaton* (RCA) is a bijective CA.

The *phase space* of a CA $(S^{\mathbb{Z}^d}, G)$ is the graph with vertices $S^{\mathbb{Z}^d}$ and two kinds of directed edges: global rule edges are pairs $(c, G(c))$ labelled by G, translation edges are pairs $(c, \varsigma_i \cdot c)$ labelled by ς_i, for all $c \in S^{\mathbb{Z}^d}$ and elementary translation ς_i. Orbits correspond to infinite G-paths in the phase space. A *periodic point*, with period $p \in \mathbb{Z}^+$, is a configuration c such that $G^p(c) = c$. A *fixpoint* is a periodic point with period 1. A *Garden-of-Eden* is a configuration c with no ancestor, *i.e.* such that $G^{-1}(c) = \emptyset$. An *ultimately periodic point*, with transient $\tau \in \mathbb{N}$ and period $p \in \mathbb{Z}^+$, is a configuration c such that $G^{p+\tau}(c) = G^\tau(c)$.

The *limit set* Λ_G of a CA $(S^{\mathbb{Z}^d}, G)$ is the non-empty translation invariant compact set $\Lambda_G = \bigcap_{i\in\mathbb{N}} \Lambda_G^{(i)}$ where $\Lambda_G^{(0)} = S^{\mathbb{Z}^d}$ and for all $i \in \mathbb{N}$, $\Lambda_G^{(i+1)} = G\left(\Lambda_G^{(i)}\right)$. The limit set consists exactly of all configurations that appear in biinfinite space-time diagrams $\Delta \in S^{\mathbb{Z}\times\mathbb{Z}^d}$ such that, for all $t \in \mathbb{Z}$, $\Delta(t+1) = G(\Delta(t))$. A CA is *nilpotent* if its limit set is a singleton. By compactness of

$S^{\mathbb{Z}^d}$, a CA $(S^{\mathbb{Z}^d}, G)$ is nilpotent if and only if there exists a uniform bound $\tau \in \mathbb{Z}^+$ such that $G^\tau(S^{\mathbb{Z}^d})$ is a singleton.

A *d-dimensional partitioned cellular automaton* (d-PCA) $\mathcal{A}$ is a triple (S, N, ψ) where S is a finite set of states, N is the neighborhood, a finite pattern of $\mathbb{Z}^d$ and $\psi : S^N \to S^N$ is the local rule of $\mathcal{A}$. The N-mixing rule $\mu_N : (S^N)^{\mathbb{Z}^d} \to (S^N)^{\mathbb{Z}^d}$ is defined, for all $c \in (S^N)^{\mathbb{Z}^d}$, for all $z \in \mathbb{Z}^d$ and for all $u \in N$, by $\mu_N(c)(z)(u) = c(z+u)(u)$. The global transition function of $\mathcal{A}$ is $\overline{\psi} \circ \mu_N$. Every PCA is a CA. Moreover, (the global transition function of) a PCA is bijective if and only if its local rule is bijective. A *reversible partitioned cellular automaton* (RPCA) is a bijective PCA.

2 Grouping Cellular Automata

The grouping quasi-order was introduced by Mazoyer and Rapaport [12] as a successful tool to classify CA according to algebraic properties [11]. However, grouping fails to capture several geometrical properties of CA that one would like to see classified by such a geometric classification. In this section, we recall the grouping quasi-order.

Grouping deals with cellular automata of dimension 1 and neighborhood $N_0 = \{-1, 0, 1\}$. In space-time diagrams of such cellular automata, a k-uple of state of a segment of k cells at time t only depends on states of $2t+k$ states at time 0, as shown on Figure 12.

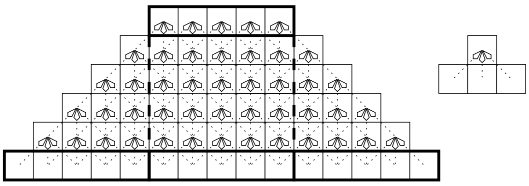


Fig. 12. dependencies in space-time diagrams for grouping

The nth iteration of the local rule is recursively defined by $f^1 = f$ and, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
f^{n+1}(x_{-n-1}, \ldots, x_{n+1}) = f(&f^n(x_{-n-1}, \ldots, x_{n-1}),\\
&f^n(x_{-n} \quad, \ldots, x_n \quad),\\
&f^n(x_{-n+1}, \ldots, x_{n+1})) \quad .
\end{aligned}
$$

Inspired by dependencies in space-time diagrams and geometrical considerations, one defines the nth grouped instance $(S, N_0, f)^n$ of a cellular automaton (S, N_0, f) by $(S, N_0, f)^n = (S^n, N_0, f^n_{\square})$ where $f^n_{\square}$ is defined for all triple of

n-uple of states by

$$f_{\square}^{n}((x_1,\ldots,x_n),(x_{n+1},\ldots,x_{2n}),(x_{2n+1},\ldots,x_{3n})) = \Big(f^n(x_1,\ldots,x_{2n+1}), \\ f^n(x_2,\ldots,x_{2n+2}), \\ \vdots \\ f^n(x_n,\ldots,x_{3n})\Big) \quad .$$

For all $n > 0$, the space-time diagrams of a cellular automaton (S, N_0, f) are in one-to-one correspondence with the space-time diagrams of its nth grouped instance $(S, N_0, f)^n$.

A cellular automaton (S, N_0, f) is *simulated* by a cellular automaton (S', N_0, f'), denoted by $(S, N_0, f) \leqslant_{\square} (S', N_0, f')$, if there exists two powers m and n such that $(S, N_0, f)^m \sqsubseteq (S', N_0, f')^n$.

Theorem 2 ([12]) *The relation $\leqslant_{\square}$ is a quasi-order relation.*

By constructing an unbounded chain of equivalence classes, the following result was proven.

Theorem 3 ([12]) *$\leqslant_{\square}$ admits no maximal element.*

3 Towards a Generalization of Grouping

Grouping can be extended in different ways: one might consider to replace the elementary simulation order (subautomaton) by another one, or one might consider to extend the set of allowed geometrical transformations to other shapes. Subsection 3.1 points out the interest of the subautomaton relation to obtain the set of all CA as an algebraic closure, subsection 3.2 shows connections between grouping and stability of a certain kind of subshifts and the need of new geometrical transformations to that extent, subsection 3.3 characterizes the most general family of space-time transformations preserving CA uniformity.

3.1 An Algebraic Characterization of Cellular Automata

Every CA is the subautomaton of a PCA. The mixing part of a PCA rule is a cartesian product of shifts, that is composition of elementary shifts; the local rule of a PCA acts as an autarkic CA. When restricting PCA to RPCA, all

RCA are generated. From there, we derive the following algebraic characterizations of CA and RCA, pointing out the use of the subautomaton relation to hide blueprint marks.

Theorem 4 *The set of d-CA is the algebraic closure of autarkic CA and elementary shifts by composition, cartesian product and subautomaton.*

PROOF. Autarkic CA and shifts being CA, the closure generates only CA.

Let (S, N, f) be a d-CA $\mathcal{A}$. Let $\varphi : S \to S^N$ map s to $(s, \ldots, s)$. By construction, $\mathcal{A}$ is a sub-automaton of the PCA $(S, N, \varphi \circ f)$ with respect to φ. The global transition function of the PCA is the composition of $\overline{\varphi \circ f}$, which is an autarkic CA, by μ_N. The N-mixing map μ_N is the product of $|N|$ shifts, each of which can be obtained as a composition of elementary shifts. ■

The restriction to RCA uses the following fact: every RCA is a subautomaton of a RPCA with neighborhood a valid neighborhood for both the RCA and its reverse. As reversibility is undecidable starting from dimension 2, there exists RCA with arbitrarily larger RPCA representation than PCA representation.

Theorem 5 *The set of d-RCA is the algebraic closure of bijective autarkic CA and elementary shifts by composition, cartesian product and subautomaton.*

PROOF. Injectivity being preserved by composition, cartesian product and subautomaton, the closure generates only RCA.

Let (S, N, f) be a d-RCA $\mathcal{A}$ with its reverse (S, N, g) a d-RCA $\mathcal{B}$ — it is always possible to choose a common neighborhood by trivially extending the local rule to the union of both neighborhood. Let $S_\bullet = S \cup \{\bullet\}$. To conclude, we introduce three RPCA $\mathcal{A}^\bullet = (S_\bullet^2, N, f_\bullet)$, $\mathcal{B}^\bullet = (S_\bullet^2, -N, g_\bullet)$ and $\mathcal{S} = (S_\bullet^2, N, h)$ such that $\mathcal{A} \sqsubseteq_\varphi \mathcal{S} \circ \mathcal{B}^\bullet \circ \mathcal{A}^\bullet$ where φ maps s to $((s, \bullet), \ldots, (s, \bullet))$.

The bijective map $f_\bullet : S_\bullet^2 \to S_\bullet^2$ is given by the following partial injective definition. For all $(s_1, \ldots, s_k) \in S^N$, let $f_\bullet((s_1, \bullet), \ldots, (s_k, \bullet)) = ((s_1, s'), \ldots, (s_k, s'))$ where $s' = f(s_1, \ldots, s_k)$.

The bijective map $g_\bullet : S_\bullet^2 \to S_\bullet^2$ is given by the following partial injective definition. For all $(s_1, \ldots, s_k) \in S^N$, let $g_\bullet((s', s_1), \ldots, (s', s_k)) = ((\bullet, s_1), \ldots, (\bullet, s_k))$ where $s' = g(s_1, \ldots, s_k)$.

The bijective map $h : S_\bullet^2 \to S_\bullet^2$ is given by the following partial injective definition. For all $(s_1, \ldots, s_k) \in S^N$, let $h((\bullet, s_1), \ldots, (\bullet, s_k)) = ((s_1, \bullet), \ldots, (s_k, \bullet))$.

Using the arguments of the proof of Theorem 4, the global transition function of every RPCA is expressible in the closure. ■

3.2 Grouping and stability of block subshifts

Note: for clarity and within this subsection only, we restrict to dimension 1.

Given a finite alphabet Σ, the set of configurations $\Sigma^{\mathbb{Z}}$ is both closed and invariant by translation. In symbolic dynamics [9], such a set is called a *full-shift* and its subsets that satisfy both properties are called *subshifts*. The image by a CA of a subshift is a subshift.

A subautomaton of a given CA $\mathcal{A}$ is always induced by a subset of states which is stable under iterations, *i.e.* a set $T \subseteq S_{\mathcal{A}}$ such that $G_{\mathcal{A}}\big(T^{\mathbb{Z}}\big) \subseteq T^{\mathbb{Z}}$.

We can establish a similar connection between the grouping relation $\leqslant_{\square}$ and a particular kind of subshifts that we call *block subshifts*. A block subshift is the set of configuration obtained by (infinite) catenation of finite words of same length from a given set. Formally, given an integer m and a set X of words of length m, the block subshift Σ_X associated to X is the set of configurations whose language is the closure of X^* by the subword operation (a subshift is characterised by the language of its configurations, see [9]). If $G_{\mathcal{A}} \sqsubseteq_{\phi} G_{\mathcal{B}}{}^{[i]}$, then $X = \phi(S_{\mathcal{A}})$ is a set of words of length i over the alphabet $S_{\mathcal{B}}$. It is straightforward to check that the block subshift Σ_X is (weakly) stable under the action of $G_{\mathcal{B}}$, *i.e.* $G_{\mathcal{B}}^i(\Sigma_X) \subseteq \Sigma_X$. Therefore any subautomaton with q states of a grouped instance of $\mathcal{B}$ is induced by a block subshift made from q words which is (weakly) stable under the action $\mathcal{B}$. The converse is false as shown by the following example: a CA can have a weakly stable block subshift made from q words without any subautomaton with q states in the corresponding grouped instance.

Example 6 *Consider $\mathcal{A}$ over state set $S_{\mathcal{A}} = \{0,1\} \times \{0,1\}$ with neighbourhood $\{0,1\}$ and local rule f defined by*

$$f\Big((a_1, b_1), (a_2, b_2)\Big) = (b_1, a_2).$$

$G_{\mathcal{A}}^2$ is the elementary right-shift CA over state set $S_{\mathcal{A}}$. So for any set X of words of length 2 over alphabet $S_{\mathcal{A}}$, the block subshift Σ_X is stable under $G_{\mathcal{A}}^2$. Now consider $G_{\mathcal{A}}{}^{[2]}$. Its only stable subset of states are of the form $Q \times Q \subseteq S_{\mathcal{A}} \times S_{\mathcal{A}}$ since $G_{\mathcal{A}}^2$ is an elementary shift. Therefore, a subautomaton of $G_{\mathcal{A}}{}^{[2]}$ must have a square number of states. ■

However, as shown by the following theorem, a larger set of geometrical transformations allows to capture all weakly stable block subshifts. This constitutes an additional motivation for the generalisation of grouping presented in the sequel.

Theorem 7 *Let i, m, q be positive integers, and $\mathcal{B}$ be a CA. The two following propositions are equivalent:*

(1) there exists a set X of q words of length m such that $G^i_{\mathcal{B}}\Big(\Sigma_X\Big) \subseteq \Sigma_X$;
(2) there exists a translation s and a CA $\mathcal{A}$ with q states which is a subautomaton of $\square^m \circ s \circ G^i_{\mathcal{B}} \circ \square^{-m}$, a CA by Theorem 1.

PROOF. First, for (2) $\Rightarrow$ (1), we suppose $G_{\mathcal{A}} \sqsubseteq_\phi \square^m \circ s \circ G^i_{\mathcal{B}} \circ \square^{-m}$ and it suffices to check that $X = \phi(S_{\mathcal{A}})$ is a set of q words of length m and that $G^i_{\mathcal{B}}\Big(\Sigma_X\Big) \subseteq \Sigma_X$ (by definition of $\sqsubseteq$ and by commutation of $G_{\mathcal{B}}$ with translations).

For (1) $\Rightarrow$ (2), we suppose (1) and consider the set E_p of configurations from Σ_X for which the catenation of words of X is aligned with position p of the lattice, formally:

$$E_p = \Big\{c \in \Sigma_X : \forall k \in \mathbb{Z}, c(km+p) \cdots c(km+p+m-1) \in X\Big\}.$$

Clearly, for any p, E_p is a closed set and $\Sigma_X = \cup_p E_p$. Moreover, $\square^m(E_0)$ is a full-shift of alphabet X. Now consider a configuration $c \in \square^m(E_0)$ whose language is X^* (a "universal" configuration, as called sometimes in the literature) and let $c' = \square^{-m}(c)$. By hypothesis, $G^i_{\mathcal{B}}(c') \in \Sigma_X$ so it belongs to some E_p. By choice of c', any $c'' \in E_0$ is obtained as the limit of some sequence $\Big(t_n(c')\Big)_n$ where each t_n is a translation of a vector multiple of m. By continuity and commutation with translations of $G_{\mathcal{B}}$ we deduce that $G_{\mathcal{B}}(c'')$ is the limit of elements of E_p so it belongs to E_p because this set is closed. Hence, $G^i_{\mathcal{B}}\Big(E_0\Big) \subseteq E_p$ and there is a suitable translation s such that $s \circ G^i_{\mathcal{B}}\Big(E_0\Big) \subseteq E_0$. From this we deduce that $\square^m(E_0)$ is a non-trivial stable full-shift of $\square^m \circ s \circ G^i_{\mathcal{B}} \circ \square^{-m}$ and the theorem follows by the discussion at the beginning of this section. ■

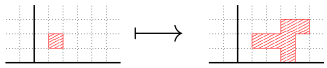
3.3 A Characterization of the Most General Geometrical Space-Time Transforms

3.3.1 Geometrical Space-Time Transforms

Grouping and classical transforms described in previous section consists of purely geometrical transforms: transforms that do not depend on the state set of the transformed CA, and thus can be applied to all CA. Such a transform maps a space-time diagram to a new space-time diagram, each space-time cell of which consists of a tuple of space-time cells of the initial diagram.

Formally, a *geometrical transform* is a pair $(k, \mathbf{\Lambda})$ where k is a positive integer and $\mathbf{\Lambda}$ maps $\mathbb{N} \times \mathbb{Z}^d$ to $\left(\mathbb{N} \times \mathbb{Z}^d\right)^k$. To help the reader visualize the transform, we will depict geometrical transforms as follows:

$$\mathbf{\Lambda} : \; \mathbb{N} \times \mathbb{Z}^d \longrightarrow \left(\mathbb{N} \times \mathbb{Z}^d\right)^k$$



The *space-time diagram transform* over a state set S by a geometrical transform $(k, \mathbf{\Lambda})$, is the map $\overline{\mathbf{\Lambda}}_S$ from $S^{\mathbb{N}\times\mathbb{Z}^d}$ to $S^{\left(\mathbb{N}\times\mathbb{Z}^d\right)^k}$ defined, for all space-time diagram $\Delta \in S^{\mathbb{N}\times\mathbb{Z}^d}$ and for all space-time position $\xi \in \mathbb{N} \times \mathbb{Z}^d$, by $\overline{\mathbf{\Lambda}}_S(\xi) = (\Delta(\lambda_1), \ldots, (\Delta(\lambda_k))$ where $\mathbf{\Lambda}(\xi) = (\lambda_1, \ldots, \lambda_k)$.

Example 8 *In 1D, the geometrical transform* $(3, \mathbf{\Lambda}^{\langle 3,4,1\rangle})$ *defined, for all* $(t,p) \in \mathbb{N} \times \mathbb{Z}$ *by* $\mathbf{\Lambda}^{\langle 3,4,1\rangle}(t,p) = ((4t, 3p+t), (4t, 3p+t+1), (4t, 3p+t+2))$ *transforms the set of all space-time diagrams* $\operatorname{Diag}\mathcal{A}$ *of every CA* $\mathcal{A}$ *into the set of all space-time diagrams* $\operatorname{Diag}\mathcal{A}^{\langle 3,4,1\rangle}$ *of a new CA* $\mathcal{A}^{\langle 3,4,1\rangle}$. *Figure 13 depicts the transform.* $\diamond$

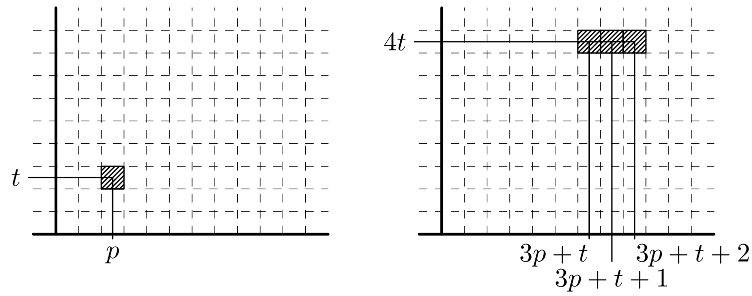


Fig. 13. Sample geometrical transform: $\left(3, \mathbf{\Lambda}^{\langle 3,4,1\rangle}\right)$

The *composition* $(k', \mathbf{\Lambda}') \circ (k, \mathbf{\Lambda})$ *of two geometrical transforms* $(k, \mathbf{\Lambda})$ and $(k', \mathbf{\Lambda}')$ is the geometrical transform $(kk', \mathbf{\Lambda}' \circ \mathbf{\Lambda})$ where, for all $\xi \in \mathbb{N} \times \mathbb{Z}^d$:

$$\left(\mathbf{\Lambda}' \circ \mathbf{\Lambda}\right)(\xi) = \left(\mathbf{\Lambda}\left(\mathbf{\Lambda}'(\xi)_1\right)_1, \ldots, \mathbf{\Lambda}\left(\mathbf{\Lambda}'(\xi)_{k'}\right)_k\right) \quad .$$

Let $\widetilde{\mathbf{\Lambda}}$ map each set of space-time cells to the set of associated space-time cells by $\mathbf{\Lambda}$:

$$\begin{aligned} \widetilde{\mathbf{\Lambda}} : 2^{\mathbb{N}\times\mathbb{Z}^d} &\longrightarrow 2^{\mathbb{N}\times\mathbb{Z}^d} \\ X &\longmapsto \bigcup_{\xi \in X} \{\mathbf{\Lambda}(\xi)_1, \ldots, \mathbf{\Lambda}(\xi)_k\} \end{aligned}$$

A *nice geometrical transform* is a geometrical transform which plays nicely with space-time diagrams and can be used to extend grouping. It should transform sets of space-time diagrams into sets of space-time diagrams and be non-trivial.

Formally, a geometrical transform $(k, \mathbf{\Lambda})$ is *nice* if it satisfies the following conditions:

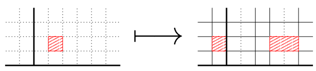
(i) for all CA $\mathcal{A}$, there exists a CA $\mathcal{B}$ such that $\overline{\mathbf{\Lambda}}_{S_{\mathcal{A}}}(\mathrm{Diag}\,\mathcal{A}) = \mathrm{Diag}\,(B)$
(ii) for all time $t \in \mathbb{N}$, $\widetilde{\mathbf{\Lambda}}(\{t+1\} \times \mathbb{Z}^d) \not\subseteq \widetilde{\mathbf{\Lambda}}(\{t\} \times \mathbb{Z}^d)$.

The set of nice geometrical transforms is closed under composition.

3.3.2 Packing, Cutting and Shifting

The classical transforms of previous section can be expressed as composition of three kinds of nice transforms, the action of which can be expressed easily in an algebraic way as global rules compositions.

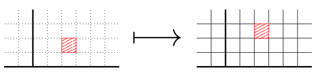
Packing. A purely spatial geometrical transform can be defined using a tiling of space to cut space regularly. The *packing transform* $\mathsf{P}_{\mathcal{P},V}$, with tiling of space $(\mathcal{P}, V)$, is defined for all $(t, p) \in \mathbb{N} \times \mathbb{Z}^d$ by:



$$\mathsf{P}_{\mathcal{P},V}\,(t,p) \;=\; \{t\} \times (\mathcal{P} + V \odot p) \quad .$$

Let $\mathcal{A}$ be a CA and $(\mathcal{P}, V)$ a tiling of space, the image of $\mathrm{Diag}\,\mathcal{A}$ by $\mathsf{P}_{\mathcal{P},V}$ is the set of space-time diagrams of the CA with global rule $\langle \mathcal{P}, V\rangle \circ G_{\mathcal{A}} \circ \langle \mathcal{P}, V\rangle^{-1}$.

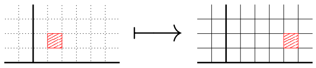
Cutting. A purely temporal geometrical transform can be defined by cutting unwanted time steps. The *cutting transform* C_T, with rate $T \in \mathbb{Z}^+$, is defined for all $(t, p) \in \mathbb{N} \times \mathbb{Z}^d$ by:



$$\mathsf{C}_T\,(t,p) \;=\; (tT, p)$$

Let $\mathcal{A}$ be a CA and $T \in \mathbb{Z}^+$ a rate, the image of $\mathrm{Diag}\,\mathcal{A}$ by C_T is the set of space-time diagrams of the CA with global rule $G_{\mathcal{A}}^T$.

Shifting. A pure translation geometrical transform can be defined by shifting space-time. The *shifting transform* S_s, with translation vector $s \in \mathbb{Z}^d$, is defined for all $(t, p) \in \mathbb{N} \times \mathbb{Z}^d$ by:



$$\mathsf{S}_s\,(t, p) \;=\; (t, p + ts)$$

Let $\mathcal{A}$ be a CA and $s \in \mathbb{Z}^d$ a translation vector, the image of $\operatorname{Diag}\mathcal{A}$ by S_s is the set of space-time diagrams of the CA with global rule $\sigma_s \circ G_{\mathcal{A}}$.

Composition. Let $(\mathcal{P}, V)$ be a tiling of space, s be a translation vector and T be a rate, the nice geometrical transform $\mathsf{PCS}_{\mathcal{P},V,T,s}$ is the composition $\mathsf{P}_{\mathcal{P},V} \circ \mathsf{S}_s \circ \mathsf{C}_T$. For all $(t, p) \in \mathbb{N} \times \mathbb{Z}^d$ it satisfies:

$$\mathsf{PCS}_{F,v,T,s}\,(t, p) = \{tT\} \times (\mathcal{P} + V \odot v + ts) \quad .$$

Let $\mathcal{A}$ be a CA, the image of $\operatorname{Diag}\mathcal{A}$ by $\mathsf{PCS}_{\mathcal{P},V,T,s}$ is the set of space-time diagrams of the CA with global rule

$$\langle \mathcal{P}, V \rangle \circ \sigma_s \circ G_{\mathcal{A}}^T \circ \langle \mathcal{P}, V \rangle^{-1} \quad .$$

Proposition 9 *The set of* PCS *transforms generated by pure* P*,* C *and* S *transforms is closed under composition.*

PROOF. Let $\mathsf{PCS}_{\mathcal{P}_1,V_1,T_1,s_1}$ and $\mathsf{PCS}_{\mathcal{P}_2,V_2,T_2,s_2}$ be two PCS transforms.

$$\mathsf{PCS}_{\mathcal{P}_2,V_2,T_2,s_2} \circ \mathsf{PCS}_{\mathcal{P}_1,V_1,T_1,s_1} = \mathsf{PCS}_{\mathcal{P}_1+(\mathcal{P}_2\odot V_1),V_2\odot V_1,T_1T_2,(s_2\odot V_1)+T_2 s_1} \qquad \blacksquare$$

3.3.3 Characterizing the Most General Transforms

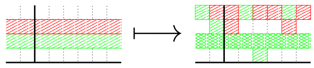
Theorem 10 *Every nice geometrical transform is a* PCS *transform.*

PROOF. Let $(k, \mathbf{\Lambda})$ be a nice geometrical transform. By definition:

(i) for all CA $\mathcal{A}$, there exists a CA $\mathcal{B}$ such that $\overline{\mathbf{\Lambda}}_{S_{\mathcal{A}}}(\operatorname{Diag}\mathcal{A}) = \operatorname{Diag}\mathcal{B}$
(ii) for all time $t \in \mathbb{N}$, $\widetilde{\mathbf{\Lambda}}(\{t+1\} \times \mathbb{Z}^d) \not\subseteq \widetilde{\mathbf{\Lambda}}(\{t\} \times \mathbb{Z}^d)$.

The proof proceeds, in 5 steps, by using (i) and (ii) for enforcing successive constraints on $\mathbf{\Lambda}$ until the PCS nature becomes clear.

Step 1. Let us first prove the following property:

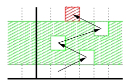


$$\forall t > 0, \forall (t', p_{t'}) \in \widetilde{\mathbf{\Lambda}}\left(\{t\} \times \mathbb{Z}^d\right) \setminus \widetilde{\mathbf{\Lambda}}\left(\{t-1\} \times \mathbb{Z}^d\right),$$
$$\exists t'' < t', \quad \{t''\} \times \mathbb{Z}^d \subseteq \widetilde{\mathbf{\Lambda}}\left(\{t-1\} \times \mathbb{Z}^d\right)$$

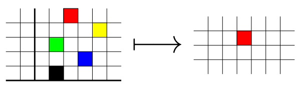
This property states that nice transforms preserve some temporal dependencies: in a transformed space-time diagram, the state of cells at time t is completely determined by the state of cells at time $t-1$.

Assume that the property is not satisfied at time t. Thus, by (ii), there exists a time t' and a sequence of spatial positions $(p_0, \ldots, p_{t'})$ such that $(t', p_{t'})$ participates to a transformed cell at time t and for all time i the cell (i, p_i) does not participate in a transformed cell at time $t-1$.

$$\begin{cases} \forall i \leqslant t', & (i, p_i) \notin \widetilde{\mathbf{\Lambda}}\left(\{t-1\} \times \mathbb{Z}^d\right) \\ & (t', p_{t'}) \in \widetilde{\mathbf{\Lambda}}\left(\{t\} \times \mathbb{Z}^d\right) \end{cases}$$

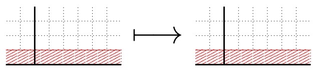


Let $\mathcal{A}$ be a CA with state set $\{\perp, 0, \ldots, t'\}$, neighborhood radius at least $\max_{i,j} |p_j - p_i|$ and such that, starting from an initial configuration uniformly equal to $\perp$ but in position p_0 where it is equal to 0, it generates a space-time diagram Δ whose $t'+1$ first configurations are uniformly equal to $\perp$ but, for each time i, the position p_i is equal to i.



The transformed coloring $\overline{\mathbf{\Lambda}}_{S_{\mathcal{A}}}(\Delta)$ is not the space-time diagram of a CA as the configuration at time $t-1$ is uniform and the configuration at time t is not: a CA cannot break such a symmetry.

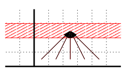
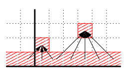
Step 2. We now show a property on initial configurations:



$$\widetilde{\mathbf{\Lambda}}\left(\{0\} \times \mathbb{Z}^d\right) = \{0\} \times \mathbb{Z}^d$$

This property states that the initial configuration of a transformed space-time diagram is obtained by a purely spatial transformation of the original initial configuration.

It follows from the fact that the image of $\operatorname{Diag} \mathcal{A}$ has to be the whole set of space-time diagram of a CA, that is, all initial configurations $\left(S_{\mathcal{A}}^k\right)^{\mathbb{Z}^d}$ should be obtained.

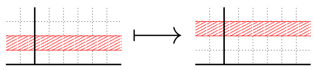
 

Assume that the property is not satisfied. Let p be a position such that there exists $(t, p') \in \widetilde{\mathbf{\Lambda}}(0, p)$ with $t' > 0$. Let $\mathcal{A}$ be the CA with two states $\{0, 1\}$ and local rule $f_{\mathcal{A}}$ constantly equal to 0. For all space-time diagrams of $\mathcal{A}$, the cell (t, p') has state 0, contradicting (i).

With a similar argument, one shows that the images of cells at time 0 are composed of disjoint cells:

$$\forall p, q, \quad \mathrm{card}\left(\widetilde{\mathbf{\Lambda}}(0, p)\right) = k \quad \wedge \quad p \neq q \Rightarrow \widetilde{\mathbf{\Lambda}}(0, p) \cap \widetilde{\mathbf{\Lambda}}(0, q) = \emptyset$$

Step 3. The previous property is extended to every time step:



$$\forall t \in \mathbb{N}, \exists t' \in \mathbb{N}, \quad \widetilde{\mathbf{\Lambda}}\left(\{t\} \times \mathbb{Z}^d\right) = \{t'\} \times \mathbb{Z}^d$$

Let t be some time step and, by **(step 1)**, let t' be such that

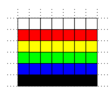
$$\widetilde{\mathbf{\Lambda}}\left(\{t\} \times \mathbb{Z}^d\right) \supseteq \{t'\} \times \mathbb{Z}^d.$$

We show first that each cell at time t contributes a same number l of cells to image cells at time t'. Formally:

$$\begin{aligned} & \exists l, \forall p, \exists i_1 < i_2 \cdots < i_l, \ (\mathbf{\Lambda}(t, p)_{i_1}, \ldots, \mathbf{\Lambda}(t, p)_{i_l}) \in \left(\{t'\} \times \mathbb{Z}^d\right)^l \\ \wedge \qquad & \forall i \notin \{i_1, \ldots, i_l\}, \ \mathbf{\Lambda}(t, p)_i \notin \{t'\} \times \mathbb{Z}^d \end{aligned}$$

Let $\mathcal{A}$ be the autarkic CA with state set $\{0, \ldots, t' + 1\}$ whose local rule f satisfies:

$$\forall i, \quad f(i) = \begin{cases} i + 1 & \text{if } i < t' + 1 \\ t' + 1 & \text{if } i = t' + 1 \end{cases}$$



Let Δ be the space-time diagram of $\mathcal{A}$ generated by the uniform configuration with state 0. By **(step 2)**, the initial configuration of $\Delta' = \overline{\mathbf{\Lambda}}_{S_{\mathcal{A}}}(\Delta)$ is also uniform and, by a symmetry argument, every configuration in Δ' is uniform. As a consequence, each cell at time t contains the same number l of component cells in state t'.

Assume that $l < k$. Let $\mathcal{A}$ be the identity CA with state set $\{0, 1\}$. Let $\mathcal{B}$ be the transformed image of $\mathcal{A}$ by $(k, \mathbf{\Lambda})$. Let Δ'_n be the space-time diagram of $\mathcal{B}$ with initial configuration uniformly equal to $(0, \ldots, 0)$ but for a ball of radius n centered in 0, this ball being filled with state $(1, \ldots, 1)$. Let Δ_n be the space-time diagram of $\mathcal{A}$ whose image by $\mathbf{\Lambda}$ is Δ'_n. By **(step 2)**, the initial configuration of Δ_n contains exactly kn^d cells with state 1. At time t', the configuration of Δ_n is equal to the configuration at time 0. By **(step 1)**, the configuration of Δ'_n at time t contains at least $\left\lceil \frac{k}{l} n^d \right\rceil$ cells with state different from $(0, \ldots, 0)$. Thus, the neighborhood radius of $\mathcal{B}$ is at least

$$r_n \geqslant \frac{\left\lceil \frac{k}{l} n^d \right\rceil^{1/d} - n}{t}$$

By hypothesis $\frac{k}{l} > 1$, thus the sequence (r_n) grows unbounded and $\mathcal{B}$ cannot exist.

Step 4. We extend the disjoint block property:

$$\forall t, \forall p, q, \quad \operatorname{card}\left(\widetilde{\mathbf{\Lambda}}(t,p)\right) = k \quad \wedge \quad p \neq q \Rightarrow \widetilde{\mathbf{\Lambda}}(t,p) \cap \widetilde{\mathbf{\Lambda}}(t,q) = \emptyset$$

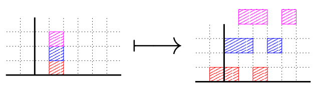
First, we show that $\operatorname{card}\left(\widetilde{\mathbf{\Lambda}}(t,p)\right) = k$: the components of each cell correspond to disjoint cells. Let $\mathcal{A}$ be the identity CA with state set $\{1, \ldots, k\}$. Let Δ be a space-time diagram of $\mathcal{A}$ such that its transformed diagram Δ' has a uniform initial configuration with state $(1, \ldots, k)$. By symmetry considerations, the configuration of Δ' at time t is uniformly filled with some state s. By **(step 1)**, s contains all possible component states 1, ..., k, thus s is a permutation of $(1, \ldots, k)$. Each component corresponds to disjoint cells.

We now show the following property:

$$\forall n, \forall t, \forall p, q, \quad (p \neq q \wedge |p - q| \leqslant n) \Rightarrow \widetilde{\mathbf{\Lambda}}(t,p) \cap \widetilde{\mathbf{\Lambda}}(t,q) = \emptyset$$

This is obtained by generalization of previous symmetry considerations. Let n be a fixed positive integer. Let $\mathcal{A}_n$ be the identity CA with state set $\left\{1, \ldots, n^d k\right\}$. Let Δ_n be the space-time diagram of $\mathcal{A}_n$ whose transformed diagram Δ'_n has a periodic initial configuration with periodic coloring the d-dimensional ball with radius n filled with states $(1, \ldots, k)$, $(k+1, \ldots, 2k)$, ..., $(n^d - 1)k + 1, \ldots, n^d k)$. By symmetry considerations, the configuration of Δ'_n at time t is periodic with a smaller period. By **(step 1)**, the periodic coloring contains all possible states 1, 2, ..., $n^d k$. Thus, all cells at distance less than or equal to n corresponds to disjoint cells.

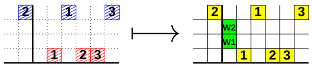
Step 5. Now that uniformity in space is achieved, next step consists in proving a property of uniformity in time:



$$\forall p, \exists s_0, T_0, \forall t, \quad \mathbf{\Lambda}(t,p) = \{tT_0\} \times \left(\pi_2\left(\mathbf{\Lambda}(0,p)\right) + ts_0\right)$$

where π_2 projects sets of space-time cells to their space components. This property states that the successive images of a given cell are regularly aligned in space-time.

Let p be a position in space. Let $(p_1, \ldots, p_k)$, $(p'_1, \ldots, p'_k)$ and t be such that $\mathbf{\Lambda}(0, p) = ((0, p_1), \ldots, (0, p_k))$ and $\mathbf{\Lambda}(1, p) = ((t, p'_1), \ldots, (t, p'_k))$.

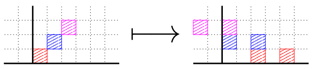


Let $\mathcal{A}$ be the CA with state set $\{\perp, 1, \ldots, k, W_1, \ldots, W_{t-1}\}$, neighborhood radius $2\max\{\|p_1\|_\infty, \ldots, \|p_k\|_\infty, \|p'_1\|_\infty, \ldots, \|p'_k\|_\infty\}$ and such that one of its space-time diagrams Δ is filled with state $\perp$ for all times between 0 and t but:

$$\begin{aligned} \Delta(0, p_1) = 1, \ldots, \Delta(0, p_k) = k, \\ \Delta(1, 0) = W_1, \ldots, \Delta(t-1, 0) = W_{t-1}, \\ \Delta(t, p'_1) = 1, \ldots, \Delta(t, p'_k) = k. \end{aligned}$$

All undefined transitions are mapped to $\perp$. Let Δ' be the transformed diagram of Δ by $(k, \mathbf{\Lambda})$. By construction, at time steps 0 and 1, the configurations of Δ' are filled with $(\perp, \ldots, \perp)$ but at position p where it is equal to $(1, \ldots, k)$. As Δ' is a space-time diagram of a CA and as its configurations at time 0 and 1 are equal, all the configurations of Δ' are equal. By construction, it implies that all the cells are uniformly shifted: $\exists s,\ p_1 = p'_1 + s \ \wedge\ \ldots\ \wedge\ p_k = p'_k + s$. The construction straightforwardly extends to all time steps.

Moreover, the previous property stands up to an elementary shift:



$$\forall i, p, \exists s_i, T_i, \forall t, \quad \mathbf{\Lambda}(t, p + t\varsigma_i) = \{tT_i\} \times (\pi_2(\mathbf{\Lambda}(0, p)) + ts_i)$$

Replay the same arguments as for previous property but considering translated cells at time 1.

Conclusion. We can now conclude that $\mathbf{\Lambda} = \mathsf{PCS}_{\pi_2(\mathbf{\Lambda}(0,0)),(s_1-s_0,\ldots,s_d-s_0),T_0,s_0}$ as elementary translations form a base for translations in $\mathbb{Z}^d$. ■

4 Axiomatics of Bulking Quasi-Orders

The basic ingredients of a bulking simulation are now clear: a set of objects with an elementary comparison relation and an algebra of (geometrical) transforms to apply on objects. Simulation is then defined by comparison up to transformation on both sides. Strong simulation is defined by comparison up to transformation on the simulator only.

In this section, the basic ingredients of bulking are formalized and properties of quasi-ordering and strong universality are established. Then, a model of bulking based on most general space-time transforms is discussed. Finally, a first model of bulking based on rectangular transforms and the subautomaton relation is introduced.

4.1 Theory of Bulking

The properties being sufficiently elementary, we choose to present bulking in a very formal way, as a first-order theory, rather than using a more classical algebraic definition. The two points of view are equivalent. Grouping being a model of bulking, it will be used to illustrate formal stuff. The formal presentation chosen here points out the elementary ingredients necessary to obtain the properties of bulking and might ease to adapt bulking to other families of objects and transformations.

Definition 11 *The* bulking *is the theory* $\mathbf{\Phi_b}$ *on the two-sorted signature*

$$\Big(\mathsf{Obj}, \mathsf{Trans}; \ \textsf{\textbf{apply}} : \mathsf{Obj} \times \mathsf{Trans} \to \mathsf{Obj}, \\ \textsf{\textbf{divide}} \subseteq \mathsf{Obj} \times \mathsf{Obj}, \\ \textsf{\textbf{combine}} : \mathsf{Trans} \times \mathsf{Trans} \to \mathsf{Trans}\Big)$$

defined by the following axioms (the meaning of which is explained below), where latin letters (x, y, $\ldots$) denotes elements of the sort Obj*, greek letters (α, β, $\ldots$) and number* 1 *denotes elements of the sort* Trans*,* x^α *denotes* $\textsf{\textbf{apply}}(x,\alpha)$*,* $x \mid y$ *denotes* $\textsf{\textbf{divide}}(x,y)$ *and* $\alpha \cdot \beta$ *denotes* $\textsf{\textbf{combine}}(\alpha,\beta)$*:*

$$(B_1) \qquad \exists 1 \forall \alpha\, (\alpha \cdot 1 = \alpha \wedge 1 \cdot \alpha = \alpha) \ \wedge \ \forall\alpha\forall\beta\forall\gamma\, ((\alpha\cdot\beta)\cdot\gamma = \alpha\cdot(\beta\cdot\gamma))$$

$$(B_2) \qquad \forall x\, (x^1 = x) \ \wedge \ \forall x \forall \alpha \forall \beta \left((x^\alpha)^\beta = x^{\alpha\cdot\beta} \right)$$

$$(B_3) \qquad \forall x\, (x \mid x) \ \wedge \ \forall x \forall y \forall z\, (((x \mid y) \wedge (y \mid z)) \to (x \mid z))$$

$$(B_4) \qquad \forall x \forall y \forall \alpha\, ((x \mid y) \to (x^\alpha \mid y^\alpha))$$

$$(B_5) \qquad \forall \alpha \forall x \exists y\, (x \mid y^\alpha)$$

$$(B_6) \qquad \forall \beta \exists \gamma \forall \alpha \exists \delta\, (\alpha \cdot \gamma = \beta \cdot \delta)$$

Definition 12 *The* simulation relation $x \preccurlyeq y$ *is syntactically defined for all* $x, y \in \mathsf{Obj}$ *by the formula* $\exists\alpha\exists\beta \left(x^\alpha \mid y^\beta\right)$*.*

Objects, transforms and relations between them can be visualized graphically by considering elements of Obj as vertices and two kind of edges: a wiggly edge labelled by an element of Trans represents **apply**, a regular edge represents the relation **divide**. The simulation relation is depicted on Figure 14.

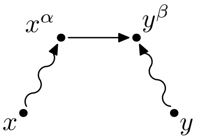


Fig. 14. Visual representation of the simulation relation

Grouping is a model of bulking where Obj is the set of global rules of 1D CA with neighborhood $\{-1, 0, 1\}$, Trans is the set of square transforms $\mathbb{N}$, $\mathbf{\mathsf{apply}}(G, n)$ is the global rule $\square^n \circ G^n \circ \square^{-n}$, $\mathbf{\mathsf{divide}}$ is the subautomaton relation $\sqsubseteq$ and $\mathbf{\mathsf{combine}}$ is the classical arithmetical product on $\mathbb{N}$.

Axioms of $\mathbf{\Phi_b}$ formalize necessary algebraic properties:

(B_1) The structure $(\mathsf{Trans}, \cdot)$ is a monoid.
(B_2) The operator **apply** is an action of the monoid $(\mathsf{Trans}, \cdot)$ on the set Obj.
(B_3) The relation **divide** is a quasi-order on Obj.
(B_4) The operator **apply** is compatible with **divide**.
(B_5) The operator **apply** preserves the diversity of objects.
(B_6) The monoid $(\mathsf{Trans}, \cdot)$ admits a strong diamond property.

The (B_5) axiom ensures transforms do not pathologically weaken the elementary simulation relation.

The (B_6) axiom might seem less natural. In fact, in the case of grouping, the square transforms commute and (B_6) could be replaced by a commutation axiom $\forall\alpha\forall\beta\,(\alpha \cdot \beta = \beta \cdot \alpha)$. Due to the shift transform, even rectangular shifted transforms do not commute but satisfy the strong diamond property of (B_6). The simulation relation is a quasi-order given only (B_{1-5}) if and only if it satisfies $\forall x\forall\alpha\forall\beta\exists\gamma\exists\delta\left(x^{\alpha\cdot\gamma} \mid x^{\beta\cdot\delta}\right)$. By letting γ depend only on x and β one can derive Theorem 14 on strongly universal objects.

Theorem 13 *The simulation relation is a quasi-order, formally:*

$$\mathbf{\Phi_b} \vdash \forall x\,(x \preccurlyeq x) \;\wedge\; \forall x\forall y\forall z\,((x \preccurlyeq y \wedge y \preccurlyeq z) \rightarrow x \preccurlyeq z)$$

PROOF. The simulation relation is reflexive: $\mathbf{\Phi_b} \vdash \forall x\,(x \preccurlyeq x)$. By combining ($B_2$) and ($B_3$), it holds $\mathbf{\Phi_b} \vdash \forall x\,(x^1 \mid x^1)$.

The simulation relation is transitive: $\mathbf{\Phi_b} \vdash \forall x\forall y\forall z\,((x \preccurlyeq y \wedge y \preccurlyeq z) \rightarrow x \preccurlyeq z)$. Let x, y, z and α, β, γ, δ be such that $x^\alpha \mid y^\beta$ and $y^\gamma \mid z^\delta$. By (B_6) there exists η and ν such that $\beta \cdot \eta = \gamma \cdot \nu$. By ($B_4$) and ($B_2$), $x^{\alpha\cdot\eta} \mid y^{\beta\cdot\eta}$ and $y^{\gamma\cdot\nu} \mid z^{\delta\cdot\nu}$. Thus, by ($B_3$), $x^{\alpha\cdot\eta} \mid z^{\delta\cdot\nu}$, implying $x \preccurlyeq z$. ■

Theorem 14 *If a strongly universal object exists then all universal objects* *are strongly universal, formally:*

$$\mathbf{\Phi_b} \vdash \exists u \forall x \exists \alpha \left(x \mid u^\alpha \right) \to \forall x \left(\forall y \left(y \preccurlyeq x \right) \to \forall y \exists \beta \left(y \mid x^\beta \right) \right)$$

PROOF. Let u be a strongly universal object: $\forall x \exists \alpha \left(x \mid u^\alpha \right)$. Let x be a universal object: $\forall y \left(y \preccurlyeq x \right)$. By universality of x, there exists α and β such that $u^\alpha \mid x^\beta$. By (B_6), there exists γ such that $\forall \eta \exists \delta \left(\eta \cdot \gamma = \alpha \cdot \delta \right)$. Let y be an object. By (B_5), there exists z such that $y \mid z^\gamma$. By strong universality of u, there exists η such that $z \mid u^\eta$. By (B_4) and (B_2), $z^\gamma \mid u^{\eta \cdot \gamma}$. Let δ be such that $\eta \cdot \gamma = \alpha \cdot \delta$. By (B_4) and (B_2), $u^{\alpha \cdot \delta} \mid x^{\beta \cdot \delta}$. Thus, by (B_3), $y \mid x^{\beta \cdot \delta}$. ■

4.2 CA Bulking: A First Attempt

Thanks to previous discussion, one might try to build a model of bulking using d-CA as sort Obj, PCS transforms as sort Trans (following Theorem 10) and the subautomaton relation $\sqsubseteq$ as elementary simulation relation (following Theorem 4). However, axiom (B_6) is not satisfied by PCS transforms. More precisely, the problem roots in the structure of the monoid of composition of tilings of space which does not admit the diamond property.

Example 15 *Let $\mathcal{P}_1 = \{0, 1\}$, $\mathcal{P}_2 = \{0, 3\}$ and $v = 2$. Let Θ be the set of all d-tilings of space. The tilings of space $\langle \mathcal{P}_1, v \rangle$ and $\langle \mathcal{P}_2, v \rangle$, depicted on Figure 15, are incompatible:* $(\Theta \circ \langle \mathcal{P}_1, v \rangle) \cap (\Theta \circ \langle \mathcal{P}_2, v \rangle) = \emptyset$. ◇

Fig. 15. Two incompatible tilings of space

Notice that the problem does not come from the composition of bases themselves (product of matrices with integer coefficients and a non-zero determinant admits a strong diamond property) but really from the geometrical shape of tilings. Intuitively, in Example 15, compositions of $\mathcal{P}_1$ always contains two consecutive elements on the extreme left whereas compositions of $\mathcal{P}_2$ contains a single element followed by a gap. Starting from dimension 2, even connected tilings lead to problem, shapes replacing gaps.

Open Problem 1 *Characterize the submonoids of composition of tilings of space that admit the diamond property.*

As we want to extend grouping, we need to select a submonoid with the strong diamond property that contains square transformations. The set of rectangular tilings constitutes an adequate commutative submonoid. One might enrich a bit this set by allowing permutations and negations of elements of the basis.

Rectangular Packing. Let $m \in (\mathbb{Z}^+)^d$ and $\tau \in \mathbb{Z}^d$ such that τ is a signed permutation of $(1, \dots, d)$. The *rectangular packing* $\tilde{\mathsf{P}}_{m,\tau}$ is the packing $\mathsf{P}_{\boxplus_m, V_\tau \odot \Box_m}$ where V_τ is the basis where $V_\tau(k)$ has all its elements equal to 0 but in position $|\tau_k|$ where it is equal to 1 if $\tau_k > 0$ and to -1 if $\tau_k < 0$.

Composition. Let (m, τ) be valid rectangular packing parameters, s be a translation vector and T be a rate, the nice geometrical transform $\tilde{\mathsf{P}}\mathsf{CS}_{m,\tau,T,s}$ is the composition $\tilde{\mathsf{P}}_{m,\tau} \circ \mathsf{S}_s \circ \mathsf{C}_T$. The set of $\tilde{\mathsf{P}}\mathsf{CS}$ transforms generated by pure $\tilde{\mathsf{P}}$, C and S transforms is closed by composition:

$$\tilde{\mathsf{P}}\mathsf{CS}_{(m'_1,\dots,m'_d),\tau',T',s'} \circ \tilde{\mathsf{P}}\mathsf{CS}_{(m_1,\dots,m_d),\tau,T,s} = \tilde{\mathsf{P}}\mathsf{CS}_{(m''_1,\dots,m''_d),\tau'',T'',s''}$$

with parameters

$$\begin{aligned} m'' &= \varrho(\tau, m, m') \quad \text{where } \varrho(\tau, m, m')_i = m_i m'_{|\tau(i)|} \\ \tau'' &= \tau' \otimes \tau \quad \text{where } (\tau' \otimes \tau)_i = \mathrm{sg}(\tau'_i) \times \tau_{|\tau'_i|} \\ T'' &= TT' \\ s'' &= (s' \odot V_\tau \odot \Box_m) + T's \end{aligned}$$

Let τ^{-1} denote the inverse of τ with respect to $\otimes$, that is such that $\tau^{-1} \otimes \tau = \mathrm{id}$ where $\mathrm{id} = (1, \dots, d)$.

To simplify notations, in the rest of the paper, $\langle m, \tau, T, s\rangle$ denotes a valid $\tilde{\mathsf{P}}\mathsf{CS}_{m,\tau,T,s}$ transform and product on this notation denotes composition. Notice that $\langle 1, \tau^{-1}, 1, 0\rangle \langle m, \tau, T, s\rangle = \langle m, \mathrm{id}, T, s\rangle$.

Composition of $\tilde{\mathsf{P}}\mathsf{CS}$ is quite symmetrical but for the shifting component.

Lemma 16 $\tilde{\mathsf{P}}\mathsf{CS}$ *transforms have the strong diamond property, that is* (B_6).

PROOF. Given a $\tilde{\mathsf{P}}\mathsf{CS}$ transform $\beta = \langle m, \tau, T, s\rangle$, let

$$\gamma = \langle \mathrm{lcm}(m)(1, \dots, 1), \mathrm{id}, \mathrm{lcm}(m)T, 0\rangle \quad .$$

For all $\alpha = \langle m', \tau', T', s'\rangle$, let $\delta = \langle \widetilde{m}, \tau' \otimes \tau^{-1}, \mathrm{lcm}(m)T', \widetilde{s}\rangle$. By definition,

$$\begin{aligned} \alpha \cdot \gamma &= \langle \mathrm{lcm}(m)m', \tau', \mathrm{lcm}(m)TT', \mathrm{lcm}(m)Ts'\rangle \\ \beta \cdot \delta &= \langle \varrho(\tau, m, \widetilde{m}), \tau', \mathrm{lcm}(m)TT', \mathrm{lcm}(m)T's + (\widetilde{s} \odot V_\tau \odot \Box_m)\rangle \end{aligned}$$

As each component of $\mathrm{lcm}(m)m'$ is a mutiple of each component of m, one can choose $\widetilde{m}$ such that $\varrho(\tau, m, \widetilde{m}) = \mathrm{lcm}(m)m'$. As each component of $\mathrm{lcm}(m)Ts'$ is a multiple of each component of m, one can choose $\widetilde{s}$ such that $\widetilde{s} \odot V_\tau \odot \Box_m = \mathrm{lcm}(m)(Ts' - T's)$. ■

Definition 17 *Let $\mathcal{A}$ and $\mathcal{B}$ be two d-CA. $\mathcal{B}$ simulates $\mathcal{A}$ injectively, denoted $\mathcal{A} \preccurlyeq_{\mathrm{i}} \mathcal{B}$, if there exists two $\tilde{\mathsf{P}}\mathsf{CS}$ transforms $\alpha = \langle m, \tau, T, s\rangle$ and $\beta = \langle m', \tau', T', s'\rangle$ such that the transform of $\mathcal{A}$ by α is a subautomaton of the transform of $\mathcal{B}$ by β. Formally,*

$$\langle \boxplus_m, V_\tau \odot \square_m \rangle \circ \sigma_s \circ G_{\mathcal{A}}^T \circ \langle \boxplus_m, V_\tau \odot \square_m \rangle^{-1} \\ \sqsubseteq \langle \boxplus_{m'}, V_{\tau'} \odot \square_{m'} \rangle \circ \sigma_{s'} \circ G_{\mathcal{B}}^{T'} \circ \langle \boxplus_{m'}, V_{\tau'} \odot \square_{m'} \rangle^{-1} \quad .$$

Theorem 18 *The set of d-CA equipped with $\tilde{\mathsf{P}}\mathsf{CS}$ transforms and the subautomaton relation $\sqsubseteq$ is a model of bulking.*

PROOF. Each axiom has to be checked:

Axiom (B_1). $(\tilde{\mathsf{P}}\mathsf{CS}, \circ)$ is a monoid with unit $\langle (1, \ldots, 1), \mathrm{id}, 1, 0\rangle$.

Axiom (B_2). By definition, applying $\tilde{\mathsf{P}}\mathsf{CS}$ transforms is an action of CA.

Axiom (B_3). The subautomaton relation is a straightforward quasi-order: $\mathcal{A} \subseteq_{\mathrm{id}} \mathcal{A}$ and if $\mathcal{A} \subseteq_\varphi \mathcal{B}$ and $\mathcal{B} \subseteq_\psi \mathcal{C}$ then $\mathcal{A} \subseteq_{\psi\circ\varphi} \mathcal{C}$.

Axiom (B_4). The subautomaton relation is compatible with $\tilde{\mathsf{P}}\mathsf{CS}$ transform by product extension of injective φ functions: if $\mathcal{A} \sqsubseteq_\varphi \mathcal{B}$ then $\mathcal{A}^{\langle m,\ldots\rangle} \sqsubseteq_\psi \mathcal{B}^{\langle m,\ldots\rangle}$ where $\psi((s_1, \ldots, s_{\prod m_i})) = (\varphi(s_1), \ldots, \varphi(s_{\prod m_i}))$.

Axiom (B_5). The application of $\tilde{\mathsf{P}}\mathsf{CS}$ transforms preserves the diversity of objects: let $\langle m, \tau, T, s\rangle$ be a $\tilde{\mathsf{P}}\mathsf{CS}$ transform and let $\mathcal{A} = (S, N, f)$ be a CA. Let $\mathcal{B} = (S, N', f')$ where $N' = V_\tau \odot \square_m \odot N$ and for all $a \in S^N$, for all $f'(V_\tau \odot \square_m \odot a)(z) = f(a)$. By construction, $\mathcal{A} \sqsubseteq_\varphi \mathcal{B}^{\langle m,\tau,T,s\rangle}$ where $\varphi(s) = (s, \ldots, s)$.

Axiom (B_6). This is Lemma 16. ■

Using Theorem 13, we conclude.

Corollary 19 (CA, $\preccurlyeq_{\mathrm{i}}$) *is a quasi-ordered set.* ■

5 CA Bulking and Universality

In this section, we investigate elementary properties of the CA bulking introduced in previous section, compare it to grouping and obtain the first results on bulking, in particular with respect to intrinsic universality. This bulking, as well as others, are studied more in depth in *Bulking II: Classifications of Cellular Automata* [4].

Definition 20 *A CA $\mathcal{U}$ is* intrinsically universal *if it strongly simulates injectively every CA: for all CA $\mathcal{A}$, there exists a $\tilde{\mathsf{P}}\mathsf{CS}$ transform α such that $\mathcal{A} \sqsubseteq \mathcal{U}^\alpha$.*

As a consequence of Theorem 14, and by existence of intrinsically universal CA [13], we can identify maximal elements.

Theorem 21 *Any maximal element of $\preccurlyeq_{\mathrm{i}}$ is intrinsically universal.* ■

As a consequence of Theorem 3, we conclude that an intrinsically universal CA has to spend some time to compute, it cannot do the computation during the time it moves the information around.

Theorem 22 *There exists no real-time intrinsically universal CA.* ■

References

[1] J. Albert, K. Čulik, II, A simple universal cellular automaton and its one-way and totalistic version, Complex Systems 1 (1) (1987) 1–16.

[2] E. R. Banks, Universality in cellular automata, in: Symposium on Switching and Automata Theory (Santa Monica, California, 1970), IEEE, 1970, pp. 194–215.

[3] C. Choffrut, K. Čulik, II, On real-time cellular automata and trellis automata, Acta Informatica 21 (4) (1984) 393–407.

[4] M. Delorme, J. Mazoyer, N. Ollinger, N. Theyssier, Bulking II: Classifications of cellular automata, *unpublished*, `oai:hal.archives-ouvertes.fr:hal-00451727` (2008).

[5] B. Durand, Z. Róka, The game of life: universality revisited, in: Cellular automata (Saissac, 1996), Kluwer Acad. Publ., Dordrecht, 1999, pp. 51–74.

[6] P. C. Fischer, Generation of primes by a one-dimensional real-time iterative array, Journal of the ACM 12 (3) (1965) 388–394.

[7] G. A. Hedlund, Endormorphisms and automorphisms of the shift dynamical system, Mathematical Systems Theory 3 (1969) 320–375.

[8] O. H. Ibarra, S. M. Kim, S. Moran, Sequential machine characterizations of trellis and cellular automata and applications, SIAM Journal on Computing 14 (2) (1985) 426–447.

[9] D. Lind, B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press, Cambridge, 1995.

[10] B. Martin, A universal cellular automaton in quasi-linear time and its s-m-n form, Theor. Comput. Sci. 123 (2) (1994) 199–237.

[11] J. Mazoyer, I. Rapaport, Additive cellular automata over Z_p and the bottom of $(\mathrm{ca}, \leq)$, in: Mathematical foundations of computer science (Brno, 1998), Springer, Berlin, 1998, pp. 834–843.

[12] J. Mazoyer, I. Rapaport, Inducing an order on cellular automata by a grouping operation, Discrete Applied Mathematics 91 (1-3) (1999) 177–196.

[13] N. Ollinger, Automates cellulaires : structures, Ph.D. thesis, École Normale Supérieure de Lyon (2002).

[14] G. Theyssier, Automates cellulaires : un modèle de complexités, Ph.D. thesis, École Normale Supérieure de Lyon (2005).